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Recurrent points and non-wandering points of graph maps ☆

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ABSTRACT

Let G be a graph and $f: G \to G$ be a continuous map. Denote by $P(f)$, $R(f)$ and $\Omega(f)$ the sets of periodic points, recurrent points and non-wandering points of f, respectively. In this paper we show that: (1) If $L = (x, y)$ is an open arc contained in an edge of G such that $\{f^m(x), f^k(y)\} \subset (x, y)$ for some $m, k \in \mathbb{N}$, then $R(f) \cap (x, y) \neq \emptyset$; (2) Any isolated point of $P(f)$ is also an isolated point of $\Omega(f)$; (3) If $x \in \Omega(f) - \Omega(f^n)$ for some $n \in \mathbb{N}$, then x is an eventually periodic point. These generalize the corresponding results in W. Huang and X. Ye (2001) [9] and J. Xiong (1983, 1986) [17,19] on interval maps or tree maps.

1. Introduction

Let (X, d) be a metric space. For any $Y \subset X$, denote by $\mathrm{Int}_X(Y)$, $\partial_X Y$ and $\overline{Y}$ the interior, the boundary and the closure of Y in X, respectively. For any $y \in Y \subset X$ and any $r > 0$, write $B(y, r) = \{x \in X\colon d(x, y) < r\}$ and $B(Y, r) = \{x \in X\colon d(x, Y) < r\}$.

Denote by $C^0(X)$ the set of all continuous maps from X to X. Let $\mathbb{N}$ be the set of all positive integers, and let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{N}$, write $\mathbb{N}_n = \{1, \ldots, n\}$. For any $f \in C^0(X)$, let f^0 be the identity map of X, and let $f^n = f \circ f^{n-1}$ be the composition map of f and f^{n-1}. A point $x \in X$ is called a *periodic point* of f with *period* n if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leqslant i < n$. The *orbit* of x under f is the set $O(x, f) \equiv \{f^n(x)\colon n \in \mathbb{Z}_+\}$. Write $\omega(x, f) = \bigcap_{i=0}^{\infty} \overline{O(f^i(x), f)}$, called the *ω-limit set* of x under f. Write $\omega(f) = \bigcup_{x \in X} \omega(x, f)$, called the *ω-limit set* of f. x is called a *recurrent point* of f if $x \in \omega(x, f)$. x is called a *non-wandering point* of f if for any neighborhood U of x there is an $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$. Denote by $P(f)$, $R(f)$ and $\Omega(f)$ the sets of periodic points, recurrent points and non-wandering points of f, respectively. From the definitions it is easy to see that $P(f) \subset R(f) \subset \omega(f) \subset \Omega(f)$. A point $x \in X$ is called an *eventually periodic point* of f if $O(x, f) \cap P(f) \neq \emptyset$. Note that x is an eventually periodic point of f if and only if the orbit $O(x, f)$ is a finite set. Denote by $EP(f)$ the set of all eventually periodic points of f. Let $h(f)$ denote the *topological entropy* of f, for the definition see [2, Chapter VIII].

A metric space X is called an *arc* (resp. an *open arc*, a *circle*) if it is homeomorphic to the interval $[0, 1]$ (resp. the open interval $(0, 1)$, the unit circle S^1). Let A be an arc and $h: [0, 1] \to A$ be a homeomorphism. The points $h(0)$ and $h(1)$ are called the *endpoints* of A and we write $\mathrm{End}(A) = \{h(0), h(1)\}$. A compact connected metric space G is called a *graph* if there are finitely many arcs $A_1, \ldots, A_n$ $(n \geqslant 1)$ in G such that $G = \bigcup_{i=1}^{n} A_i$ and $A_i \cap A_j = \mathrm{End}(A_i) \cap \mathrm{End}(A_j)$ for all $1 \leqslant i < j \leqslant n$. A graph T is called a *tree* if it contains no circle. A continuous map from a graph (resp. a tree, a circle, an interval) to itself is called a *graph map* (resp. a *tree map*, a *circle map*, an *interval map*).

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Let G be a given graph. Take a metric d on G such that, for any $x \in G$ and any $r > 0$, the open ball $B(x,r) \equiv \{y \in G: d(y,x) < r\}$ is connected. For any finite set S, let $|S|$ denote the number of elements of S. For any $x \in G$, write $val_G(x) = \lim_{r\to+0} |\partial_G B(x,r)|$, which is called the *valence* of x in G. x is called a *branching point* (resp. an *endpoint*) of G if $val_G(x) > 2$ (resp. $val_G(x) = 1$). Denote by $\mathrm{End}(G)$ and $\mathrm{Br}(G)$ the sets of endpoints and branching points of G, respectively. Take a finite subset $V(G)$ of G containing $\mathrm{End}(G) \cup \mathrm{Br}(G)$ such that, for any connected component E of $G - V(G)$, the closure $\overline{E}$ is an arc. Every point in $V(G)$ is called a *vertex* of G, and every connected component of $G - V(G)$ is called an *edge*. For any edge E of G and any two points $a, b \in \overline{E}$, we denote by $[a,b]_E$ (or simply $[a,b]$ if there is no confusion) the smallest connected closed subset of $\overline{E}$ containing $\{a,b\}$, and we write $(a,b] = [b,a) = [a,b] - \{a\}$ and $(a,b) = (a,b] - \{b\}$.

In the study of dynamical systems, periodic points, recurrent points, ω-limit points, non-wandering points and the sets of these points play an important role. Graphs (containing intervals, circles and trees) are 1-dimensional spaces. In the study of dynamical systems of graph maps, one can obtain more, finer and interesting results. It is shown that $\Omega_2(f) \equiv \Omega(f|\Omega(f)) = \overline{P(f)}$ for any interval map f, and $\Omega(f|\Omega(f)) = \overline{R(f)}$ for any circle map f, see [2,14,15,17] and [6], respectively. Hence the depths of interval maps and circle maps are at most 2. In [16], this result was generalized to continuous maps of n-od with zero being a fixed point. In [20,21], Ye studied properties of non-wandering points of tree maps and graph maps and showed that the depths of tree maps and graph maps are at most 3. In [13], it is shown that $\Omega_2(f) = \overline{R(f)}$ also holds for any graph map f. Block [1] studied interval maps with finite non-wandering set. Coven and Nitecki [7] and Huang and Ye [9] studied the difference between $\Omega(f)$ and $\Omega(f^n)$ for interval map and tree map f, respectively. In [5], Coven and Hedlund showed that $\overline{P(f)} = \overline{R(f)}$ for interval map f. Recently, Mai and Shao [11] showed that $\overline{R(f)} = R(f) \cup \overline{P(f)}$ for graph map f. In [3,4], Blokh constructed the "spectral" decomposition of the sets $\overline{P(f)}, \omega(f)$ and $\Omega(f)$ for any graph map f, and obtained a series of applications of the "spectral" decomposition. In [8], Hric and Malek gave a full topological characterization of the ω-limit sets of graph maps and showed that basic sets have similar properties as in the case of the closed interval. In [18], Xiong showed that every point of $\Omega(f) - \overline{P(f)}$ is one-side isolated in $\Omega(f)$ for any interval map f.

In this paper we will study recurrent points and non-wandering points of graph maps. Our main results are following theorems, which generalize the corresponding results in [9,17,19] on interval maps or tree maps.

Theorem 2.8. *Let $f: G \to G$ be a graph map, and $L = (x, y)$ be an open arc contained in an edge of G. If there exist $m, k \in \mathbb{N}$ such that $\{f^m(x), f^k(y)\} \subset (x, y)$, then $R(f) \cap (x, y) \neq \emptyset$.*

Theorem 3.7. *Let $f: G \to G$ be a graph map. Then any isolated point of $P(f)$ is also an isolated point of $\Omega(f)$.*

Theorem 4.2. *Let $f: G \to G$ be a graph map and $x \in G$. If $x \in \Omega(f) - \Omega(f^n)$ for some $n > 1$, then x is an eventually periodic point of f.*

2. Recurrent points of graph maps in given arcs

From Lemma IV.9 of [2] or from the unique lemma of [17] one can directly derive the following proposition, which gives a sufficient condition for an interval map to have a periodic point in a given subinterval.

Proposition A. *Let I be a compact interval, $f \in C^0(I)$ and $[x, y] \subset I$. If there exist $m, k \in \mathbb{N}$ such that $\{f^m(x), f^k(y)\} \subset (x, y)$, then $P(f) \cap (x, y) \neq \emptyset$.*

In this section we will generalize the study to graph maps. We will present a theorem, which gives a sufficient condition for a graph map to have recurrent points in a given arc. First we raise a lemma, which is simple but useful.

Lemma 2.1. *Let $f: G \to G$ be a graph map, $L = (a, b)$ be an open arc contained in an edge of G, and $c \in (a, b)$. If $c = f^k(b)$ for some $k \in \mathbb{N}$ and $P(f) \cap [a, b] = \emptyset$, then $[a, c] \subset \bigcup_{i=1}^{\infty} f^{ik}([c, b])$.*

Proof. Write $W = \bigcup_{i=1}^{\infty} f^{ik}([c, b])$. Then W is a connected set containing $c = f^k(b)$. If there exists $u \in [a, c] - W$, then, since $P(f^k) \cap [a, b] = P(f) \cap [a, b] = \emptyset$, we have $W \subset (u, b)$ and $f^k(\overline{W}) \subset \overline{W} \subset [u, b]$. But this will yield $P(f) \cap [a, b] \supset P(f^k) \cap \overline{W} \neq \emptyset$ and lead to a contradiction. Thus we must have $[a, c] \subset W$. □

The following three lemmas are also useful in this paper, which are results obtained in [12,13].

Lemma 2.2. *(See [12, Corollary 1].) Let $f: G \to G$ be a graph map. Then $x \in \Omega(f)$ if and only if there exist points $x_k \to x$ and positive integers $n_k \to \infty$ such that $f^{n_k}(x_k) = x$.*

Lemma 2.3. *(See [12, Proposition 3].) Let $f: G \to G$ be a graph map. If $x \in \Omega(f) - \overline{R(f)}$, then there exist $\varepsilon > 0$ with $B(x, \varepsilon) \cap V(G) \subset \{x\}$ and $y \in \partial_G B(x, \varepsilon)$ such that $\overline{B(x, \varepsilon)} \cap (\bigcup_{n=1}^{\infty} f^n([x, y])) = \emptyset$.*

Lemma 2.4. *(See [13, Proposition 2.3].) Let $f : G \to G$ be a graph map, $L = (a, b)$ be a connected component of $G - \overline{R(f)} - V(G)$, and $x \in (a, b) \cap \Omega(f)$. If there exists $y \in (x, b)$ such that $x \notin \bigcup_{n=1}^{\infty} f^n([x, y])$, then $f^m(w) \notin [x, w)$ for any $w \in (x, b]$ and any $m \in \mathbb{N}$.*

From the above three lemmas we can obtain

Proposition 2.5. *Let $f : G \to G$ be a graph map, $L = (a, b)$ be a connected component of $G - \overline{R(f)} - V(G)$, and $x \in (a, b) \cap \Omega(f)$. Then there exists $y \in (a, b) - \{x\}$ such that $[x, y] \cap (\bigcup_{n=1}^{\infty} f^n([x, b])) = \emptyset$ with $y \in (x, b)$, or $[y, x] \cap (\bigcup_{n=1}^{\infty} f^n([a, x])) = \emptyset$ with $y \in (a, x)$.*

Proof. Let $x_k \to x$ be as in Lemma 2.2. We may assume that $(x_1, x_2, \ldots)$ is a monotonic sequence in (a, b). If $\{x_1, x_2, \ldots\} \subset (a, x)$, then by Lemma 2.3 there is $y \in (x, b)$ such that $[x, y] \cap (\bigcup_{n=1}^{\infty} f^n([x, y])) = \emptyset$, and by Lemma 2.4 we get $[x, y] \cap (\bigcup_{n=1}^{\infty} f^n((y, b])) = \emptyset$, these lead to $[x, y] \cap (\bigcup_{n=1}^{\infty} f^n([x, b])) = \emptyset$. Similarly, if $\{x_1, x_2, \ldots\} \subset (x, b)$, then there is $y \in (a, x)$ such that $[y, x] \cap (\bigcup_{n=1}^{\infty} f^n([a, x])) = \emptyset$. $\square$

Lemma 2.6. *Let $f : G \to G$ be a graph map, $L = (a, b)$ be an open arc contained in a connected component of $G - \overline{R(f)} - V(G)$, and $c \in (a, b)$. If $b = f^m(a)$ and $c = f^k(b)$ for some $m, k \in \mathbb{N}$, then there exists $j \in \mathbb{N}$ such that $f^{m+jk}(c) \in (c, b)$.*

Proof. Assume on the contrary that $f^{m+jk}(c) \notin (c, b)$ for all $j \in \mathbb{N}$. Noting that $[a, b] \cap P(f) = \emptyset$, from $f^{m+k}(a) = c$ and $f^{m+k}(c) \notin (c, b)$ we see that there exists $a_1 \in (a, c)$ such that $f^{m+k}([a, a_1]) \supset [c, b]$ with $f^{m+k}(a_1) = b$. Also, from $f^{m+2k}(a_1) = c$ and $f^{m+2k}(c) \notin (c, b)$ we see that there exists $a_2 \in (a_1, c)$ such that $f^{m+2k}(a_2) = b$. By induction, we get points $a_1, a_2, \ldots \in (a, c)$ satisfying $f^{m+ik}(a_i) = b$ and $a_{i+1} \in (a_i, c)$ for all $i \in \mathbb{N}$. Let $u = \lim_{i\to\infty} a_i$. Then $u \in (a, c]$. By Lemma 2.1, there exists $v \in (c, b]$ such that $u \in O(v, f^k) \subset O(v, f)$.

Note that $f^{m+(i+1)k}([a_i, a_{i+1}])$ is a connected set containing both $c = f^{m+(i+1)k}(a_i)$ and $b = f^{m+(i+1)k}(a_{i+1})$. If there exists $i \in \mathbb{N}$ such that $v \notin f^{m+(i+1)k}([a_i, a_{i+1}])$, then we have $v \in (c, b)$ and $[a_i, a_{i+1}] \subset (a, c) \subset f^{m+(i+1)k}([a_i, a_{i+1}])$, and there exists an arc $A \subset [a_i, a_{i+1}]$ such that $f^{m+(i+1)k}(A) = [a_i, a_{i+1}]$. But this contradicts $P(f) \cap [a_i, a_{i+1}] = \emptyset$. Thus for each $i \in \mathbb{N}$ there exists $u_i \in [a_i, a_{i+1}]$ such that $f^{m+(i+1)k}(u_i) = v$, which implies $u \in O(u_i, f)$. Hence, by Lemma 2.2 we have $u \in \Omega(f)$, and by Proposition 2.5 we get $u \notin O(v, f)$. But this still lead to a contradiction, and Lemma 2.6 is proved. $\square$

Lemma 2.7. *Let $f : G \to G$ be a graph map, and $L = (a, b)$ be an open arc contained in an edge of G. If there exist $m, k \in \mathbb{N}$ such that $f^m(a) = b$ and $f^k(b) \in (a, b)$, then $R(f) \cap (a, b) \neq \emptyset$.*

Proof. Assume on the contrary that $R(f) \cap (a, b) = \emptyset$. Then $\overline{R(f)} \cap [a, b] = \emptyset$ since $f^{m+k}(a) = f^k(b) \in (a, b)$ and $(a, b) \cap V(G) = \emptyset$. Write $a_1 = a, a_2 = b$ and $a_3 = c = f^k(b)$. By Lemma 2.6, there exist points $\{a_i\colon i \in \mathbb{N}\} \subset [a, b]$ such that $a_{i+1} \in O(a_i, f)$ and $a_{i+2} \in (a_i, a_{i+1})$ for all $i \in \mathbb{N}$. Clearly, both $(a_1, a_3, a_5, \ldots)$ and $(a_2, a_4, a_6, \ldots)$ are strictly monotonic sequences. Let $x = \lim_{i\to\infty} a_{2i+1}$. Then $x \in \omega(a, f) \subset \omega(f) \subset \Omega(f)$, and $a_{2i+2} \in (x, a_{2i})$ for all $i \in \mathbb{N}$. On the other hand, by Lemma 2.4 we should have $a_{2i+2} \notin [x, a_{2i})$. This contradiction shows that $R(f) \cap (a, b) \neq \emptyset$ must hold. $\square$

We now give the main result of this section and its proof.

Theorem 2.8. *Let $f : G \to G$ be a graph map, and $L = (x, y)$ be an open arc contained in an edge of G. If there exist $m, k \in \mathbb{N}$ such that $\{f^m(x), f^k(y)\} \subset (x, y)$, then $R(f) \cap (x, y) \neq \emptyset$.*

Proof. If $P(f) \cap (x, y) \neq \emptyset$, then the theorem holds since $P(f) \subset R(f)$. We now assume that $P(f) \cap (x, y) = \emptyset$. Write $w = f^k(y)$. If $y \in f^m([x, w])$, then there exists a point $a \in (x, w)$ such that $f^m(a) = y$ and $f^k(y) = w \in (a, y)$, and by Lemma 2.7 we get $R(f) \cap (x, y) \supset R(f) \cap (a, y) \neq \emptyset$. If $y \notin f^m([x, w])$, then $f^m(u) \in (u, y)$ for all $u \in [x, w]$. In this case we have $f^k(y) = w$ and $f^m(w) \in (w, y)$, and by Lemma 2.7 we also get $R(f) \cap (x, y) \supset R(f) \cap (w, y) \neq \emptyset$. $\square$

Corollary 2.9. *Let $f : G \to G$ be a graph map, C be a circle in G, and $v \in C$. If $C \cap \mathrm{Br}(G) = \{v\}$, and $f^m(v) \in C$ for some $m \in \mathbb{N}$, then $R(f) \cap C \neq \emptyset$.*

Proof. It suffices to consider the case that $f^m(v) \in C - \{v\}$. Choose an $r > 0$ such that $f^m(\overline{B(v, r)}) \cap \overline{B(v, r)} = \emptyset$. Then $f^m(\overline{B(v, r)}) \subset C - \overline{B(v, r)}$. Let x and y be the two points in $C \cap \partial_G B(v, r)$. Take the set $V(G)$ of vertexes of G such that $V(G) \cap C = \{v, x, y\}$. Let $L = (x, y) = C - \overline{B(v, r)}$. Then L is an edge of G and $f^m(\{x, y\}) \subset L$. By Theorem 2.8, we have $R(f) \cap C \supset R(f) \cap L \neq \emptyset$. $\square$

Remark 2.10. It is easy to show that, in Theorem 2.8, if $G = T$ is a tree, then the conclusion $R(f) \cap (x, y) \neq \emptyset$ can be strengthened to be $P(f) \cap (x, y) \neq \emptyset$. However, if G is not a tree, then the conclusion in the theorem cannot be strengthened so. For example, let $G = C$ be a circle, and $f : C \to C$ be an irrational rotation. Then for any open arc (x, y) in C there exist $m, k \in \mathbb{N}$ such that $\{f^m(x), f^k(y)\} \subset (x, y)$, but we do not have $P(f) \cap (x, y) \neq \emptyset$ since $P(f) = \emptyset$.

3. Isolated periodic points of graph maps

In [19], Xiong proved the following:

Proposition B. *(See [19, Proposition 2], also [2, Proposition IV.35].) Let $f : I \to I$ be an interval map. Then any isolated point of $P(f)$ is also an isolated point of $\Omega(f)$.*

In this section we will generalize this result to graph maps. First we recall a well-known lemma. This lemma is trivial, but it is very useful in the study of interval maps.

Lemma 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. If there exist a compact interval $A \subset \mathbb{R}$ and $m \in \mathbb{N}$ such that $f^m(A) \subset A$ or $f^m(A) \supset A$, then $P(f) \cap A \neq \emptyset$.*

The following lemma is also trivial, which is a generalization of Lemma 3.1.

Lemma 3.2. *Let X be a topological space and $f : X \to X$ be a continuous map. If there exist an arc $A \subset X$ and $m \in \mathbb{N}$ such that one of the following two conditions holds*:

(1) $f^m(A) \subset A$;
(2) $f^m(A) \supset A$, *and there exist a subarc L of A such that $f^m(L) = A$*;

then $P(f) \cap A \neq \emptyset$.

In Lemma 3.2, the second condition "$f^m(A) \supset A$, and there exist a subarc L of A such that $f^m(L) = A$" cannot be simplified to "$f^m(A) \supset A$" only. In fact, we have:

Example 3.3. For any $t \in \mathbb{R}$, write $w_t = e^{2\pi \mathbf{i} t}$, where $\mathbf{i} = \sqrt{-1}$. Then w_t is a point in the complex plane $\mathbb{C}$. Let $S^1 = \{w_t\colon t \in \mathbb{R}\}$. Then S^1 is the unit circle. Let $r > 0$ be a given irrational number. Define a homeomorphism $\psi : S^1 \to S^1$ by $\psi(w_t) = w_{t+r}$ for any $t \in \mathbb{R}$. Let $C = \{2w_t\colon t \in \mathbb{R}\}$. Then C is a circle in $\mathbb{C}$ with radius 2. We can define a continuous surjection $h : C \to S^1$ satisfying the following three conditions:

(1) for any $n \in \mathbb{Z}_+$, $h^{-1}(w_{-nr})$ is an arc in C with length 2^{-n};
(2) for any $w \in S^1 - \{w_{-nr}\colon n \in \mathbb{Z}_+\}$, $h^{-1}(w)$ contains exactly one point;
(3) for any connected subset W of S^1, $h^{-1}(W)$ is a connected subset of C.

And then we can construct a continuous map $f : C \to C$ such that

(4) for any $n \in \mathbb{N}$, $f(h^{-1}(w_{-nr})) = h^{-1}(w_{r-nr})$, and $f|h^{-1}(w_{-nr})$ is an injection;
(5) for any $w \in S^1 - \{w_{-nr}\colon n \in \mathbb{N}\}$, $f(h^{-1}(w)) = h^{-1}(\psi(w))$;
(6) $hf = \psi h$, that is, h is a topological semi-conjugacy from f to ψ.

Let $L = h^{-1}(w_0)$. Then L is an arc in C, f sends L to the point $h^{-1}(w_r)$, and $f|(C - L)$ is an injection. Let $A = \overline{C - L}$. Then A is an arc and $f(A) = C \supset A$, but we do not have $P(f) \cap A \neq \emptyset$ since f itself is a circle map having no periodic points.

For the structure of general graph maps having no periodic points, see [10].

Lemma 3.4. *Let $f : G \to G$ be a graph map. Then any isolated point of $P(f)$ is also an isolated point of $R(f)$.*

Proof. Let x be a periodic point of f with period λ. Write $g = f^\lambda$. Then $P(f) = P(g)$, $R(f) = R(g)$ and $g(x) = x$. If x is isolated in $P(f)$ but x is not isolated in $R(f)$, then there exists an arc $A = [x, y]$ in G such that $(x, y] \cap (P(g) \cup V(G)) = \emptyset$ and $(x, y) \cap R(g) \neq \emptyset$. For the convenience of statement, we may assume that $[x, y] = [0, 1]$ is the unit interval with $x = 0$ and $y = 1$. Take a point $w \in R(g) \cap (0, 1)$. Since $w \notin P(f)$, it is easy to see that there exist integers $m > j > 0$ and points $\{w_0, w_j, w_m\} \subset O(w, g) \cap (0, 1)$ such that $w_j = g^j(w_0)$, $w_m = g^m(w_0)$, and $0 < w_0 < w_m < w_j < 1$.

Take points $a \in (0, w_0)$ and $b \in (w_0, w_m)$ such that $[a, b]$, $g^j([a, b])$ and $g^m([a, b])$ are mutually disjoint closed intervals contained in $(0, 1)$. Write $I_0 = [a, b]$, $I_j = g^j(I_0)$, and $I_m = g^m(I_0)$. Then $w_n \in I_n$ for $n \in \{0, j, m\}$.

Since $O(w, g) \cap I_m \supset g^m(O(w, g) \cap I_0)$ is an infinite set contained in $R(g) - P(g)$, there exist $\{u, v\} \subset O(w, g) \cap I_m$ and $k \in \mathbb{N}$ such that $v = g^k(u) < u$. Let $Y = \bigcup_{n=0}^{\infty} g^{nk}([v, u])$. Then Y is a connected set. If $x = 0 \notin Y$, then, since $P(g) \cap (0, u] = \emptyset$, we will have $Y \subset (0, u]$, and $(u, g^k(u), g^{2k}(u), \ldots)$ will be a strictly decreasing sequence in $(0, u]$. But this contradicts that $u \in R(g^k) = R(g)$. Thus we have $0 \in Y$, that is, there exists $\gamma \in \mathbb{N}$ such that $0 \in g^{\gamma k}([v, u])$.

Since $[v,u] \subset I_m$, there exist $\{a_0, b_0\} \subset I_0$ with $a_0 < b_0$ and $\{a_j, b_j\} \subset I_j$ with $a_j < b_j$ such that $g^m([a_0,b_0]) = g^{m-j}([a_j,b_j]) = [v,u]$. Since $u \in R(g^k)$ and $x = 0 < a_0 < b_0 < v < u < a_j < b_j < 1$, there is an integer $\beta > \gamma$ such that $g^{\beta k}(u) \in (v, a_j)$. Let $W = g^{\beta k}([v,u])$. Then W is a connected set containing both $x = 0$ and $g^{\beta k}(u)$, and hence $W \cap \{b_0, a_j\} \neq \emptyset$. Let $r = \max(g^{-\beta k}(0) \cap [v,u])$, and $t = \min(g^{-\beta k}(\{b_0, a_j\}) \cap [r,u])$. Then $v < r < t < u$. If $g^{\beta k}(t) = b_0$, then there exists $s \in (r,t)$ such that $g^{\beta k}([s,t]) = [a_0,b_0]$ and hence $g^{\beta k+m}([s,t]) = [v,u]$. By Lemma 3.2, we have $P(g) \cap [v,u] \neq \emptyset$. But this will lead to a contradiction. If $g^{\beta k}(t) = a_j$, then there exists $s \in (r,t)$ such that $g^{\beta k}([s,t]) = [a_j,b_j]$ and hence $g^{\beta k+m-j}([s,t]) = [v,u]$. By Lemma 3.2, we also have $P(g) \cap [v,u] \neq \emptyset$, and this still lead to a contradiction. Thus each isolated point x of $P(f)$ must be isolated in $R(f)$. Lemma 3.4 is proved. □

Lemma 3.5. *(See [13, Proposition 2.5].) Let $f : G \to G$ be a graph map, $L = (x_0, b)$ be a connected component of $G - \overline{R(f)} - V(G)$, and $x_1, x_2, \ldots, x_n$ be n points in $\Omega(f) \cap L$ with $n > 1$ and $x_i \in (x_{i-1}, x_{i+1})$ for $1 \leqslant i \leqslant n-1$. If there exist $y \in G$ and $z \in (x_0, x_1)$ such that $y \in \bigcap_{i=1}^n O(x_i, f)$ and $x_1 \in O(z, f)$, then $val_G(y) \geqslant n$.*

Corollary 3.6. *Let $f : G \to G$ be a graph map, $L = (u, v)$ be a connected component of $G - \overline{R(f)} - V(G)$, and $w_1, w_2, \ldots, w_{2n-1}$ be $2n-1$ distinct points in $\Omega(f) \cap L$ with $n > 1$. If there exists $y \in G$ such that $y \in \bigcap_{i=1}^{2n-1} O(w_i, f)$, then $val_G(y) \geqslant n$.*

Proof. We may assume that $w_i \in (w_{i-1}, w_{i+1})$ for $2 \leqslant i \leqslant 2n-2$. Since $w_n \in \Omega(f) - R(f)$, by Lemma 2.2, there exists $z \in (w_{n-1}, w_{n+1}) - \{w_n\}$ such that $w_n \in O(z, f)$. For $i \in \mathbb{N}_n$, put $x_i = w_{n+1-i}$ if $z \in (w_n, w_{n+1})$, and put $x_i = w_{i+n-1}$ if $z \in (w_{n-1}, w_n)$. Then from Lemma 3.5 we get $val_G(y) \geqslant n$. □

We now give the main result of this section and its proof, which is a generalization of the above Proposition B.

Theorem 3.7. *Let $f : G \to G$ be a graph map. Then any isolated point of $P(f)$ is also an isolated point of $\Omega(f)$.*

Proof. Let x be an isolated point of $P(f)$. Then by Lemma 3.4 there exists a connected open neighborhood W of x in G such that $(W - \{x\}) \cap (R(f) \cup V(G)) = \emptyset$. Let m be the period of x under f, and let $n = \max\{val_G(v)\colon v \in G\}$. Then $f^m(x) = x$, and there exists a neighborhood $U \subset W$ of x such that $f^{im}(U) \subset W$ for all $i \in \mathbb{N}_{(2n+1)n}$. Assume that x is not isolated in $\Omega(f)$. Then there is an arc $A = [x,u] \subset U$ such that $(x,u) \cap \Omega(f)$ is an infinite set. Let $w_1, w_2, \ldots, w_{2n+1}$ be $2n+1$ points in $(x,u) \cap \Omega(f)$. If $x \in \bigcap_{i=1}^{2n+1} O(w_i, f)$, then by Corollary 3.6 we will have $val_G(x) \geqslant n+1$. But this contradicts the definition of n. If there exists $k \in \mathbb{N}_{2n+1}$ such that $x \notin O(w_k, f)$, put $Q = \{f^{im}(w_k)\colon i \in \mathbb{N}_{(2n+1)n}\}$. Then there is an arc $A' = [x,u'] \subset W$ such that $(x,u') \cap Q$ contains at least $2n+1$ points. Let $v_1, v_2, \ldots, v_{2n+1}$ be $2n+1$ points in $(x,u') \cap Q$. Write $y = f^{(2n+2)n}(w_k)$. Then we have $y \in \bigcap_{i=1}^{2n+1} O(v_i, f)$, and by Corollary 3.6 we will get $val_G(y) \geqslant n+1$. But this still lead to a contradiction. Thus the isolated point x of $P(f)$ must be isolated in $\Omega(f)$. Theorem 3.7 is proved. □

4. Eventually periodic non-wandering points of graph maps

For tree maps, Huang and Ye [9] proved the following:

Theorem C. *(See [9, Theorem 1.1].) Let T be a tree, $f \in C^0(T)$ and $x \in \Omega(f)$. Then*

(1) *$x \in \Omega(f^n)$ for each $n \in \mathbb{N}$ if $O(x, f)$ is infinite;*
(2) *$\Omega(f) = \Omega(f^n)$ for each $n \in \mathbb{N}$ if $h(f) = 0$.*

In this section we will generalize the first conclusion of this theorem to graph maps, and will give an example to show that the second conclusion of this theorem is not true for general graph maps. Note that the first conclusion of Theorem C has an equivalent statement: If $f : T \to T$ is a tree map and $x \in \Omega(f) - \Omega(f^n)$ for some $n > 1$, then the orbit $O(x, f)$ is a finite set, that is, x is an eventually periodic point of f.

Let G be a graph. A tree T in G is called a *topologically maximal subtree* of G if $\mathrm{Br}(T) = \mathrm{Br}(G)$ and $val_T(v) = val_G(v)$ for all $v \in \mathrm{Br}(T)$. Define a function

$$\xi(G) = \sum\{val_G(v) - 2\colon v \in \mathrm{Br}(G)\},$$

called the *total branching number* of G. For any connected set $X \subset G$, let $N(X, G)$ denote the number of connected components of $X - \mathrm{Br}(G)$. We first present a lemma, which gives the supremum of numbers of boundary points of connected sets in G.

Lemma 4.1. *Let G be a graph, T be a topologically maximal subtree of G which contains no endpoint of G, and $X \subset G$ be a connected set. Then*

$$|\partial_G X| \leqslant |\partial_G T| = \left|\mathrm{End}(T)\right| = \xi(G) + 2.$$

Proof. By induction for the number of branching points of the tree T it is easy to verify that $|\mathrm{End}(T)| = \xi(T) + 2$. From the definition of topologically maximal subtree T we see that $\xi(T) = \xi(G)$ and $\partial_G T \subset \mathrm{End}(T)$. On the other hand, since T contains no endpoint of G, we have $\mathrm{End}(T) \subset \partial_G T$. Thus $\partial_G T = \mathrm{End}(T)$, and hence $|\partial_G T| = |\mathrm{End}(T)| = \xi(G) + 2$.

It remains to show that $|\partial_G X| \leqslant \xi(G) + 2$. We may consider only the case that X contains more than one point. Write $c = \sup\{d(x, y):\ x, y \in X\}$. Since X is connected, for any edge E of G, $\partial_G X \cap E$ contains at most two points. Thus $\partial_G X$ is a finite set. Take an $\varepsilon \in (0, c/9]$ such that $(B(y, 2\varepsilon) - \{y\}) \cap \mathrm{Br}(G) = \emptyset$ for any $y \in \partial_G X \cup \mathrm{End}(G)$. Let $X_1 = X - \bigcup\{B(y, \varepsilon):\ y \in (\overline{X} - X) \cup \mathrm{End}(G)\}$. Then X_1 is a connected closed subset of G, $X_1 \cap \mathrm{End}(G) = \emptyset$, and $|\partial_G X| \leqslant |\partial_G X_1|$. Again, choose a finite set $S \subset X_1 - \mathrm{Br}(G)$ and an $\delta \in (0, c/9]$ such that $X_1 - S$ is a connected set containing no circle and $B(S, 2\delta) \cap \mathrm{Br}(G) = \emptyset$. Put $X_2 = X_1 - B(S, \delta)$. Then X_2 is a subtree of G and $|\partial_G X_1| \leqslant |\partial_G X_2|$. Note that if X_1 itself is a tree then we can take $S = \emptyset$ and $X_2 = X_1 - B(\emptyset, \delta) = X_1 - \emptyset = X_1$. Finally, we can take a sequence of subtrees $X_2 \subset X_3 \subset \cdots \subset X_m$ of G for some $m \geqslant 2$ such that

(1) $N(X_{i+1}, G) = N(X_i, G) + 1$ and $|\partial_G X_i| \leqslant |\partial_G X_{i+1}|$, for $2 \leqslant i \leqslant m - 1$,
(2) $X_i \cap \mathrm{End}(G) = \emptyset$, for $2 \leqslant i \leqslant m$,
(3) X_m is a topologically maximal subtree of G.

Therefore, we get $|\partial_G X| \leqslant |\partial_G X_1| \leqslant |\partial_G X_2| \leqslant \cdots \leqslant |\partial_G X_m| = \xi(G) + 2$. Lemma 4.1 is proved. □

We now give the main result of this section and its proof, which is a generalization of the first conclusion of the above Theorem C.

Theorem 4.2. *Let $f : G \to G$ be a graph map and $x \in G$. If $x \in \Omega(f) - \Omega(f^n)$ for some $n > 1$, then x is an eventually periodic point of f.*

Proof. Since $x \in \Omega(f)$ but $x \notin \Omega(f^n)$, from Lemma 2.2 we see that there exist $\{x_1, x_2, \ldots\} \subset G$, $\lambda \in \mathbb{N}_{n-1}$ and $\{\beta_1 < \beta_2 < \beta_3 < \cdots\} \subset \mathbb{N}$ such that $\lim_{i\to\infty} x_i = x$ and $f^{n\beta_i+\lambda}(x_i) = x$ for all $i \in \mathbb{N}$. Write $k_i = n\beta_i + \lambda$. Let

$$S = \left\{ j \in \mathbb{N}_n:\ x \in \bigcap_{m=1}^{\infty} \left(\bigcup_{\beta=1}^{\infty} f^{n\beta+j\lambda}\big(B(x, 1/m)\big) \right) \right\}.$$

Then $1 \in S$ and $n \notin S$. Let $\mu = \max(S)$. Then $1 \leqslant \mu \leqslant n - 1$, and there exist $\{y_1, y_2, \ldots\} \subset G$ and $\{\gamma_1 < \gamma_2 < \gamma_3 < \cdots\} \subset \mathbb{N}$ such that $\lim_{i\to\infty} y_i = x$ and $f^{n\gamma_i+\mu\lambda}(y_i) = x$ for all $i \in \mathbb{N}$. Write $m_i = n\gamma_i + \mu\lambda$.

Since $\mu + 1 \notin S$, there exists a connected open neighborhood $U = B(x, \varepsilon)$ such that $x \notin \bigcup_{\beta=1}^{\infty} f^{n\beta+(\mu+1)\lambda}(U)$. Without loss of generality, we may assume that the above points $\{x_1, x_2, \ldots\}$ and $\{y_1, y_2, \ldots\}$ are all in U. So, for any $i > j \geqslant 1$, we have

$$f^{k_i-k_j}(x) = f^{k_i-k_j}\big(f^{k_j}(x_j)\big) = f^{k_i}(x_j) \in f^{k_i}(U) \subset \mathrm{Int}_G\big(f^{k_i}(U)\big) \cup \partial_G\big(f^{k_i}(U)\big).$$

If $f^{k_i-k_j}(x) \in \mathrm{Int}_G(f^{k_i}(U))$ for some $i > j \geqslant 1$, then there exists a sufficiently large integer $p \in \mathbb{N}$ such that $f^{k_i-k_j}(y_p) \in \mathrm{Int}_G(f^{k_i}(U)) \subset f^{k_i}(U)$. Hence, there is a point $w \in U$ such that $f^{k_i-k_j}(y_p) = f^{k_i}(w)$, which implies that

$$x = f^{m_p}(y_p) = f^{m_p-k_i+k_j}\big(f^{k_i-k_j}(y_p)\big) = f^{m_p+k_j}(w) \in f^{m_p+k_j}(U) = f^{n(\gamma_p+\beta_j)+(\mu+1)\lambda}(U) \subset \bigcup_{\beta=1}^{\infty} f^{n\beta+(\mu+1)\lambda}(U).$$

But this will lead to a contradiction. Thus, for any $i > j \geqslant 1$, we must have $f^{k_i-k_j}(x) \in \partial_G(f^{k_i}(U))$. Since $f^{k_i}(U)$ is a connected subset of G, by Lemma 4.1, its boundary $\partial_G(f^{k_i}(U))$ contains at most $\xi(G) + 2$ points. Let $q = \xi(G) + 4$. Then there exist $1 \leqslant j < i < q$ such that $f^{k_q-k_i}(x) = f^{k_q-k_j}(x) \in \partial_G(f^{k_q}(U))$. This means that x is an eventually periodic point of f. Theorem 4.2 is proved. □

We now give an example to show that the second conclusion of Theorem C is not true for general graph maps.

Example 4.3. For any $n \geqslant 2$, there exists a graph map $f : G \to G$ such that $h(f) = 0$ but $\Omega(f) \neq \Omega(f^n)$.

Proof. For $k \in \{0, 1, \ldots, n - 1\}$, let $S_k = \{(x, y, k(1 - x)) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$. Then S_k is a (topological) circle. Let $A = \{(x, 0, 0) \in \mathbb{R}^3:\ -1 \leqslant x \leqslant 0\}$. Then A is a line segment. Put $G = A \cup (\bigcup_{k=0}^{n-1} S_k)$. Then G is a graph in $\mathbb{R}^3$. Define a continuous map $f : G \to G$ by

$$f(x,y,z)=\begin{cases}(x,y,(k+1)(1-x))\in S_{k+1}, & \text{if } (x,y,z)\in S_k \text{ and } 1\leqslant k\leqslant n-2;\\ (x,y,0)\in S_0, & \text{if } (x,y,z)\in S_{n-1};\\ (\cos 2\theta,\sin 2\theta,1-\cos 2\theta)\in S_1, & \text{if } (x,y,z)=(\cos\theta,\sin\theta,0)\in S_0 \text{ with } \theta\in[0,\pi];\\ (1,0,0)\in S_1, & \text{if } (x,y,z)\in S_0 \text{ with } y\leqslant 0;\\ (\cos(1+x)\pi,\sin(1+x)\pi,0)\in S_0, & \text{if } (x,y,z)=(x,0,0)\in A.\end{cases}$$

Then we have $\Omega(f)=\{(1,0,0),(-1,0,0)\}\neq\Omega(f^n)=\{(1,0,0)\}$. On the other hand, by [2, Theorem VIII.6], we have $h(f)=h(f|\Omega(f))$. Since $\Omega(f)$ contains only two points, $h(f|\Omega(f))=0$. Thus we have $h(f)=0$. $\square$

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